Mathematical symbols as epistemic actions – an extended mind perspective

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Abstract
Why are mathematical models so surprisingly efficient in the sciences? In this paper, adopt an extended mind perspective on human mathematical abilities. I review evidence from animal, infant and neuropsychology studies that the human brain contains several specialized neural circuits which can be co-opted to solve complex mathematical problems. However, mathematics has emergent properties which cannot be reduced to these cognitive subsystems. In particular, experimental evidence suggests that our evolved number sense is only capable of representing approximate quantities. Active externalism, a cognitive mechanism proposed by various authors, including Andy Clark and Merlin Donald, allows humans to overcome these cognitive limitations by performing epistemic actions in the world that could not be performed in the mind alone. I discuss the extensive use of external symbols in the history of early modern European mathematics, and demonstrate that this has led to an increasing efficiency of mathematics as epistemic tool in the sciences. This externalization of mathematical symbols can be traced back in the archaeological record to at least 20 000 years ago. Wider implications of externalist and evolutionary approaches to understanding mathematical cognition are discussed.

1. Introduction

In the philosophy of mathematics, questions on the nature of mathematics fall naturally into three different categories. First, there are ontological issues, which have been popular since the beginning of the 20th century: what are mathematical objects, where do they reside, and do they exist independently of the human brain? Second, there are foundational questions, which were fashionable in the 19th and first part of the 20th century: which axioms do we need as a basis for a mathematical system, is it possible to construct a consistent and complete mathematical system? Third, there is the epistemology of mathematics, which has perhaps received the least attention of all. It asks how mathematical knowledge is possible – i.e. how people are able to have mathematical knowledge, and how it enables us to extend our knowledge in other sciences as well. The main reason why this question was neglected was the tendency to believe that mathematics does not rest on any psychological foundations at all (computationalism). This computational approach, as advocated by Turing and others, equated mathematics with logic. Indeed, the computer revolution which brought us so-called 'thinking machines' seemed to show that mathematics rests on the application of a strict set of laws of thought (logic). The main problem with this approach is that logic and mathematics parted ways (Decock 2005), so that we are still left with the question why humans can create mathematical knowledge at all. Since the late 1990s, new insights from cognitive science and psychology have emerged, which are of potential interest to the epistemology of mathematics. Neuroscientists such as Stanislas Dehaene (1997) and psychologists like Lakoff and Núñez (2000) consider mathematics as a form of embodied cognition: human mathematicians, equipped with human brains come up with, share and elaborate upon mathematical ideas.

My aim in this talk is to take an embodied cognitive approach to examine how and why mathematical concepts are sources of creative thought for a many of the sciences. I will first briefly review the applicability of mathematics in the sciences, showing that mathematics is a versatile (although not indispensable) tool for creative scientific thought. Next, I will examine the roots of our mathematical abilities through recent studies in animal cognition, developmental psychology and neuroscience. Although these abilities are robust and adaptive, I demonstrate that they cannot account for the complexities of mathematical thought. I argue that active externalism, the consistent attempts of mathematicians to formulate mathematical operations through external symbols, is the main cause for the abstraction and flexibility of mathematical thought. In particular, externalism allows mathematicians to perform operations that would be impossible to create in the mind alone, such as subtracting a large number from a smaller one. I illustrate this with examples in the history of early modern European mathematics. Finally, I ask how cognitive and externalist approaches can help elucidate why mathematical systems have the epistemic properties they have.

2. The applicability of mathematics in the sciences

One of the most intriguing features of mathematics is its applicability to the sciences. Its 'unreasonable effectiveness' in the sciences has puzzled philosophers of mathematics (e.g. Wigner, 1960) for decades. Mathematics can be likened to a microscope, because it enables us to discover processes in nature which would not have been possible to unravel without it. An elegant illustration is the application of simple off-the-shelf algebra by William Harvey (17th century), who calculated the total volume of blood pumped per hour through the human heart (*Exercitatio Anatimica De Motu Cordis et Sanguinis In Animabilius, An Anatomical Disquisition on the Motion of the Heart and Blood in Animals*, first published in 1628). From this, he inferred that there must be some small vessels that conveyed the blood from outgoing arteries to the returning veins, even though he was unable to observe these small vessels. His prediction, formulated solely on the basis of anatomical knowledge and mathematical calculations, was spectacularly confirmed more than half a century later, when Marcello Malpighi detected capillary vessels under his microscope. This example from biology illustrates the enormous epistemic value of mathematics combined with deductive reasoning and empirical observations (Cohen 2004). In physics, the effectiveness of mathematics is aptly demonstrated by Albert Einstein's shifting point of view on mathematics: whereas the young Einstein considered it as an empty tool, ancillary to physics, he later identified it as a principle source of scientific creativity in combination with empirical observations (Corry 1998). The observation of mathematics' efficiency in the sciences has induced Hilary Putnam and Willard V.O. Quine to the ontological commitment that mathematical entities have an existence independent of human observers. This indispensability argument is a pragmatic argument for mathematical realism, the general view that mathematical entities exist outside of human mind and culture. It goes as follows: if you believe that scientific theories such as quantum mechanics or general relativity are true (i.e. if you have a realist position in respect to these theorems) you must also grant mathematics an ontological status independent of human reasoning, since mathematics is indispensable for these and other disparate theories.

Because the early successes of mathematics in the sciences set high standards for later work in both biology and physics, the indispensability of mathematics for science can become easily overstated. Drawing on extensive mathematics to formulate theories and predictions has even become a must rather than an epistemic tool in itself. I illustrate this through an example in biology, the handicap principle, first formulated by Amotz Zahavi in 1975. Briefly stated, the handicap principle states that high-quality animals may benefit from investing a part of their advantage in advertising their quality to potential mates, competitors or predators, by taking on a handicap, in a way that inferior individuals would not be able to do, because for them the investment would be too high. For example, a healthy gazelle struts (makes high leaps) in front of a charging lion instead of running away, to convince him that pursuing it would be a loss of time. The lion will desist, and will instead attack another gazelle that does not strut and is likely to be weaker. Although the model could elegantly explain the existence of seemingly redundant and costly traits and behaviours, including the peacock's tail or alarm calls in birds, it was not accepted by the majority of biologists, who distrusted it because it was not stated in a mathematical form. As Zahavi (2003: 860) later put it ‘For some reason that I cannot understand, logical models expressed verbally are often rejected as being “intuitive”’. In 1990, Grafen finally made the Handicap Principle acceptable for biologists by reformulating it in a game theoretical model. However, Grafen stated that his conclusions were the same as in Zahavi’s original paper, and he admitted that the model was simple enough to be stated verbally. Nevertheless, according to Zahavi ‘biologists remained unimpressed by the logic of the verbal model, and accepted the handicap principle only when expressed in a complex mathematical model, which I and probably many other ethologists do not understand’ (Zahavi 2003: 860). This example illustrates that although mathematics has become essential in most branches of science, scientific investigation is just as well possible without mathematical formulations. Indeed, Charles Darwin (1859), the founding father of evolutionary biology, unfolded his principle of natural selection through the selective retention of fit variations solely in verbal format, through rich metaphors and fastidious empirical observations. Nevertheless, even if mathematics is not indispensable in science, we must still explain why mathematics has the epistemic virtues it has, and why it is an unexpected source of creativity. I will now look at the possible evolutionary roots of our abilities for mathematical thought, to examine if these can explain the computational power of mathematics.

3. Mathematical concepts are more than mathematical cognition

A converging body of empirical evidence from infant development, animal behaviour and neuro-imaging suggests that human mathematical abilities are more than a cultural invention, and that they are rooted in cognitive evolution. Scientists from different disciplines have argued that humans and other vertebrates share an evolved ‘number sense’, a specialized neural mechanism for detecting numerosities in the environment. The evolutionary roots of numerical abilities are perhaps best illustrated by animals which spontaneously use numerical cues when making adaptive decisions. For example, lionesses decide whether or not to attack an intruding group based on a comparison of the number of unfamiliar roaring individuals they hear and the number of members of the pride present (McComb et al. 1994). The same results have been obtained for chimpanzees, who only decide to attack intruders when at least three group members are present (Wilson et al. 2000). Under experimental conditions, shoaling fish prefer a shoal with more members over one with fewer members (Barber & Wright 2001). Animals also use numerical cues to guide their foraging decisions: when presented with two patches of food-items both rhesus monkeys (Hauser et al. 2000) and red-backed salamanders (Uller et al. 2003) go for the larger quantity.

Experiments based on preferential looking time with human infants have demonstrated their numerical competence prior to language or explicit teaching. Newborns can discriminate between sets of 2 and 3 items (Antell & Keating 1983). Five-month-olds can predict the outcome of simple addition and subtraction events (e.g. 1 + 1 = 2 and not 1) (Wynn 1992). Even in experiments where non-numerical variables such as total surface area, density and contour length are controlled for, this numerical ability remains robust. For example, eight-month-olds can detect changes in numerosity when surface area is kept constant, but fail to detect a twofold increase in constant cumulative surface area when numerosity varies (Brannon et al. 2004). Thus, rather than relying on continuous variables to detect numerosities, infants seem better at detecting numerosities than other variables. Finally, the ability of infants to detect numerosities is not restricted to the visual modality: they can also keep track of events (such as the number of times a puppet jumps, Wynn 1996), and sequences of sounds (Lipton & Spelke 2003). Taken together, this evidence from animal and infant studies seems to suggest that the number sense is an evolved capacity, which we share with all vertebrates. Neuro-imaging studies indicate that the number sense depends on a highly specialized neural circuit, located in the horizontal banks of the intraparietal cortex

(HIPS). This area of the brain shows increased activation when subjects discriminate numerosities. Its activation is strongest in approximate calculation (Dehaene et al. 1999), and has been reliably dissociated from other non-numerical tasks. For example, the fMRI study by Eger et al. (2003) found that the presentation of an Arabic digit, say '3' activates the HIPS, whereas a similar-looking letter, say 'A' does not. Even when presented in an auditory format, the word 'two' elicits activation in the HIPS, whereas the word 'red' does not. Thus, this area is activated in the most elementary and cross-modal recognition of numerosities. In a set of single cell studies Nieder and colleagues (Nieder et al. 2002; Nieder & Miller 2003) presented rhesus monkeys with pairs of slides with dots which varied in size, shape or numerosity. They found individual neurons that responded only to changes in number, while remaining insensitive to changes in shape or size. Neuropsychological studies suggest that our number sense can be selectively impaired or spared. For example, Varley et al. (2005) found that the ability to perform exact multiple digit-calculations remains intact in patients with extreme aphasia, suggesting that complex calculations can be performed without relying on linguistic skills.

While our evolved number sense is robust across modalities and is apparently useful in adaptive decisions, it is as yet unclear how this cognitive adaptation can account for the vast proliferation and complexity of cultural mathematical concepts. As Karen Wynn aptly put it:

> "One achievement unique to the human species is the development of cultural knowledge that is elaborated and enriched over many successive generations. The formal system of mathematics, the historical development of which can be traced back for thousands of years, is an excellent example of this kind of knowledge. More work needs to be done to clarify just how our unlearned core of numerical competence, one which we hold in common with other species, interacts with other cognitive capacities to allow the development of this elegant and complex system". (Karen Wynn 1998: 124)

Several psychological studies with adults illustrate that our ability to represent numerosities resembles more a logarithmic ruler than a linear number line. When adults are prevented from counting, their performance grows imprecise as the numerosity increases (Cordes et al. 2001), a phenomenon known as Weber's Law. This law states that the discriminability of two magnitudes (e.g. two solid objects of different weight or two sounds of different pitch) is determined by the ratio of the objective magnitudes. For instance, a weight difference of 5 grams between two objects of 5 and 10 grams is more observable than the difference between 55 and 60 grams. Thus, adults find it harder to discriminate between 6 and 8 than between 2 and 3, even if the absolute difference between 2 and 3 is smaller than that between 6 and 8. Human infants, likewise, can only discriminate between large numerosities when the ratio difference is large. For example, 6-month-olds successfully discriminate between 8 and 16 dots, but not between 8 and 12 dots (Xu & Spelke 2000). In fact, once counting and other cultural tools are out of the picture, human numerical performance is very comparable to that of non-human animals. Brannon & Terrace (2002) asked college students to perform a numerical comparison task between pairs of Arabic numerals and then trained monkeys to perform the same task. There was a striking resemblance between human and monkey performance. In both species, the response time (latency) and accuracy show similar distance and size effects. This suggests that human adults (as well as rhesus monkeys) convert sets of discrete stimuli (the Arabic numerals) to a continuous magnitude representation before the comparison process itself takes place. Indeed, in human cultures without exact number words, such as two Amazonian cultures Pirahã (Gordon, 2004) and Mundurukú (Pica et al. 2004), the ability to reason about exact numerosities is restricted to very small numbers. Conversely, despite extensive training, no non-human animal has yet been able to learn exact natural number representations. One long-term training programme extending over 20 years involved teaching Ai, a female chimpanzee, to understand and produce Arabic digits (Biro &

Matsuzawa 2001). She was trained to type the number of items she saw on a keyboard. Despite extensive training, Ai could only manage to count up to 9. She apparently learned to associate numerosities with cardinal values by association, but never generalized this to the counting procedure that children master with ease. In the initial training, she assumed that ‘2’ meant ‘more than one’, but eventually she learned to apply ‘1’, ‘2’and ‘3’ correctly. Instead of generalizing this procedure to numbers greater than 3, as human children do, she evidently went on to assume that ‘3’ meant more than two, which brought her competence at assigning ‘4’ correctly to chance performance. Because she failed to generalize to the counting procedure, she needed the same span of time to learn the remaining integers, which is not observed in children, who make a ‘leap forward’ and improve vastly in their counting-abilities once they reach number 3 (Hauser & Spelke 2004). These lines of evidence combined suggest that although our mathematical abilities build upon an evolved number sense which we share with other animals, they are clearly more than that.

4. Cultural mathematical concepts and cognitive fluidity

Why is it that mathematical concepts are more than the cognitive abilities that lie at the basis of it? One possibility may be that humans are able to overcome their cognitive limitations by combining several evolved neural circuits. Cognitive archaeologist Steven Mithen (1996) has explored this possibility, arguing that the human mind exhibits a *cognitive fluidity*. Unlike other animals, the human mind combines evolved cognitive domains to apply these to different computational problems, thus vastly extending the range of possible solutions to any given computational problem. The strongest evidence for cognitive fluidity is found in religious concepts, which cut across domains, as in totemism (humans descending from animals), speaking trees or eating mountains.

Could cognitive fluidity explain human cultural mathematical concepts? Several brain-imaging studies could be interpreted as evidence for this hypothesis. For example, Dehaene and colleagues (1999) found that humans rely on linguistic neural circuits as well as non-linguistic ones when they perform calculations. Exact calculations elicit greater activation in the linguistic circuits, showing that humans rely on verbally memorized routines when performing complex calculations. Sandrini, Rossini & Miniussi (2004) found that a momentary disruption of the left intraparietal lobule, involved in the representation of body parts, disrupts adults’ performance in a numerical comparison task, showing that initial finger-counting may continue to play its part in adult numerical cognition. Finally, Zorzi et al. (2002) found that patients with hemispatial neglect, i.e. people who cannot attend to anything on either their left or right sight, have an impaired mental number line as well: when asked to bisect a number interval without counting, such as 11-19, their estimations systematically shifted overtly to the left or right, e.g. 17 as the bisection of 11-19. This study supports the view that non-numerical spatial mental representations play an important role in conceptualizing numbers on a mental number line.

If many neural circuits are co-opted in simple calculations and number comparisons, it is not surprising that this is also true for more complex mathematical tasks, such as algebra. FMRI studies (e.g. Qin et al. 2004) found that people use different brain circuits when they solve equations: the intraparietal sulci, which have been implicated in brain imaging studies involving number (e.g. Dehaene et al. 1999); the anterior cingulate cortex, which is otherwise active when people reflect on other people’s mental states (Frith & Frith 2001); and the posterior parietal cortex, which is normally activated in visuospatial tasks, including spatial working memory and attention orienting (Simon et al. 2002). One study investigating the difference between mathematical strategies in normal and in precocious teenagers found that the latter deployed spatial skills to solve algebraic problems by diagramming important relationships apparent in the problem (Dark & Benbow 1991). Thus, solving equations depends on a successful co-optation of several brain areas which are normally involved in ecologically relevant tasks such as detecting numerosities, and assessing spatial relationships. However, this same study casts doubt on the cognitive fluidity hypothesis because learning equations results in changes in cerebral blood flow after a learning period. In other words, the brain rewires its synaptic connections in response to the task-demands. Thus, we cannot explain cultural mathematical concepts solely as a result of the combination of cognitive systems, since learning those results in permanent synaptic changes in a few well-restricted areas of the brain.

5. Human culture and active externalism

This examination of the cognitive basis of our abilities to engage in mathematical thought demonstrates that these evolved abilities are but a prerequisite. They are by no means sufficient to explain the complexities of human mathematical abilities. Even if we accept Mithen's suggestion for a cognitively fluid mind, we are still far from explaining how the calculus or algebra could have emerged. The solution to this puzzle may seem unproblematic: why don't we just attribute it to cultural evolution? As Restivo (1992) and other social constructivists argued, mathematical representations are social, cultural structures. However, culture in itself cannot be invoked as a final causal mechanism to explain aspects of human behaviour for the following reasons. Science has an implicit ontological commitment to a materialist monism. In order to engage in science, we assume (a) that the real world is material, and that there are no autonomous subject-free ideas, and (b) that every event in the world is the outcome of prior causal events or laws; there is no uncaused cause (Bunge 2004). In other words, scientific investigation explains events or properties of the world by looking for causal mechanisms, which can be traced to material, tangible elements in the world. Thoughts can accordingly be seen as electrophysiological communication between neurons via synaptic connections and neurotransmitters, and cultural transmission can be properly conceptualized as mediated through speech (sound waves which travel through space) or visual media (light falling on the retina of the observer) (Sperber 1996). The principle of universal causation (also termed ubiquity determinism), which states that every event satisfies some law(s), is a necessary foundation for science. Once science admits the possibility of an event without cause, it has abandoned its own mission. In practice it is impossible to rule out that such events exist. Proofs of causality or its absence are, as Hume (1739-1740) already noted, impossible: causality is inferred by our own mind and cannot be directly observed. However, science is ontologically committed to seek causal explanations, since that constitutes its very aim. The principle of determinacy can thus be preferred over the uncaused cause, since it does not place dogmatic bounds on the scope of scientific enquiry and explanation (Hodgson 2002: 276). These premises do not imply that all causal processes are deterministic in the narrow sense: many a causal mechanism arises from random processes on lower levels, e.g. in quantum chemistry, a reaction of the type A + B -> C is constituted of microchemical reactions, each of which with a probability that C will emerge (Bunge 2004: 195-196). Nor do they imply that science can realize Laplace's dream: since scientific theories deal with open, complex systems, measurements and theories cannot be all-encompassing since they would have to describe the whole universe. However, it does mean that we cannot simply evoke culture as an ultimate causal mechanism to explain the complexities of human mathematical thought.

Like all other events in the world, human culture can be explained as the outcome of specific causal mechanisms, which can be described in purely materialistic terms. Although non-human animals engage in sophisticated forms of social learning, only our species has cumulative cultural traditions. Unlike other primates, our cultural systems build and elaborate upon the inventions of previous generations, which has resulted in a vast proliferation of artefact types, customs, religious beliefs and social systems that vary across cultures. Why only humans can benefit from this ratchet effect of cultural evolution is not entirely clear. One proposal, formulated by Clark & Chalmers (1998) and Donald (1991) is that human cognition benefits from an active externalism. As Andy Clark (2001: 134) put it:

"…one large jump or discontinuity in human cognitive evolution involves the distinctive way human brains repeatedly create and exploit various species of cognitive technology so as to expand and re-shape the space of human reason. We – more than any other creature on the planet – deploy non-biological elements (instruments, media, notation) to *complement* our basic biological modes of processing, creating extended cognitive systems…"

Rather than performing computations in the brain alone, humans heavily rely on external media to delegate parts of these computations to the external world. External memory devices such as books or electronically stored documents, instruments such as computers or the nautical slide ruler, or even simply pen and paper, all serve to extend our computations beyond the brain. There are at least three possible ways in which externalism can play a role in enhancing computational abilities: as artificial memory storage (Donald 1991), as epistemic act (Kirsh & Maglio 1994), and as anchor for non-intuitive ideas (Mithen 2000). I discuss each of these possibilities and their relations to mathematics.

5.1. Externalism as artificial memory system

Many cognitive tasks require a considerable degree of memory storage. Learning a new language, handicraft, or discipline puts a strain on our memory abilities. Cultural transmission would be limited to what an individual human brain could memorize were it not for external media. Clearly, the use of external media enables us to accumulate information beyond the scope of the individual memory. Since humans have invented how to store information externally, it has become uncertain where the locus of human memory is. Clark & Chalmers (1998) have therefore famously argued that externally stored information that is reliably accessible could be rightly called beliefs, even if they are not stored in our brains *per se*. For example, the writings in a notebook by an Alzheimer patient, who makes extensive use of this device to store and retrieve facts, could be rightly regarded as part of his beliefs. Donald (1991) proposes that the invention of visual symbolic systems, starting with engraved bones and visual art, and culminating in the invention of writing, constitutes a key event in human cognitive evolution. Storing and retrieving information that is non-intuitive and complex can perhaps only take place when it is underpinned by external storage. d'Errico (1998) shows through a morphological analysis of late Palaeolithic artefacts that mathematical knowledge was stored before writing was invented. In some cases, the artefacts were probably used as simple tally-sticks, a way of making exact representations of numerosities by putting the items to be counted into a one-to-one correspondence with notches carved on bone or stone objects. For example, rib of a woolly rhinoceros excavated at Solutré and dated 22 000-17 000 BP shows hundreds of fine lines, made with the same point, on the concave side. Since the notches do not exhibit an intentional spatial distribution, they may have been accumulated over time. Micro-wear analysis of the blunted point of the rib indicates that it could have been a digging stick. The notches, consequently, could have served to keep count of items that were retrieved by use of this digging stick. Some Palaeolithic objects, like the La Marche antler dated at 17 000 -11 000 found in a cave in Western France (d'Errico 1995) show variations in spatial distribution and morphology which were apparently intentionally made. These external symbols may have provided an artificial storage of information. Indeed, Marshack (1991) has argued that several incised Palaeolithic bones served as calendrical notation systems. Contemporary mathematicians, like their prehistoric predecessors, heavily rely on externally stored information that they can retrieve at will. Their beliefs are distributed in their brain, and in books and other external media. Without external media, mathematics as we know it could not exist.

5.2. Externalism as epistemic act

Working with external symbolic media does not just provide artificial memory systems; it often constitutes an epistemic act in itself. Some actions performed by use of external media solve problems more easily and reliably than if they would be solved in the mind alone. Take the computer game Tetris: physically rotating the two-dimensional blocks by means of a keyboard in order to fit them in the slots proves far more efficient than mental rotation (Kirsh & Maglio 1994). The use of external media makes computational solutions possible that could not have been reached without them. In mathematics, a positional system such as the Hindu numerals renders multiplications with large numbers easy and transparent, while multiplication in non-positional systems such as the Roman numerals is a far more daunting task. From the history of mathematics, we can infer that the absence or presence of symbols that represent operations influences the degree of abstraction within specific mathematical systems.

During the 16th-17th century, a fundamental shift in the perception of science itself enabled European scientists to create radical turnovers such as the transformation from Ptolemaic to Keplerian astronomy. Medieval mathematicians believed that mathematics had deep roots in classical Greece, which were sometimes lost, and sometimes rediscovered, but there always remained an unchanging body of knowledge which could at best be preserved and at worst corrupted by contemporary scholars. Gradually, however, they came to realize that mathematicians could elaborate and even improve upon this body of knowledge. The English 17th century historian of mathematics John Wallis, for example, not only wrote that the Muslims had developed and improved classical texts, but proudly stated the further achievements of his contemporaries (Stedall 2001). European mathematicians thus became increasingly aware that they themselves created the levels of abstraction in mathematics. This realization led to many innovations such as the consistent use of negative numbers, the invention of imaginary numbers and of non-Euclidean geometry. Take *i*, which symbolizes an operation that cannot actually be performed – namely extracting a square root from a negative number. By denoting this operation it becomes possible to integrate roots from negative numbers in operations, thus vastly extending the scope of mathematics. The operation which is impossible to perform in the brain can be performed in the world, basically by inventing a symbol to represent it. Only European mathematics has consistently aimed at representing such opaque operations by a constant set of symbols (Restivo 1992). Indian mathematicians, for example, used the symbol '0' both for zero and for the unknown in equations (Joseph 1990). Such ambiguities may have limited the degree of abstraction in Indian mathematics. Chinese algebra provides another example. Since the beginning of the Common Era, Chinese mathematicians had developed elaborate matrix methods to solve simultaneous equations. However, Chinese algebra textbooks never attempted to give an abstract formulation of a general rule, but presented examples that served as paradigms to solve similar problems (Chemla 2003). One of the reasons for this lack of abstraction is that Chinese algebra was not performed by means of a symbolic notation system, but merely by placing counting rods in rows and columns. These rods were less useful to express general abstract rules other than actual calculations, which preserved the concreteness of Chinese mathematics (Joseph 1990).

5.3. Externalism as anchor of autonomous mathematical thought

External media can provide anchors for thoughts that are difficult to understand or represent (Mithen 2000). Without external symbols, such thoughts would not survive long in the competition for attention and cognitive resources that characterizes cultural evolution. Indeed, theoretical models that examine cultural representations from an epidemiological perspective (e.g. Sperber 1985) predict that concepts that are hard to learn and hard to represent are quickly outcompeted in favour of intuitive or attention-grabbing ideas. These predictions have been experimentally confirmed in a variety of cultural settings. For example, story recall experiments illustrate that both Hindu (Barrett 1998) and Christian (Barrett & Keil 1996) college students do not intuitively think about their respective gods as their theologies require. Christians have difficulties representing God as an omniscient, omnipresent being and distort stories about him to fit intuitive expectations they have about normal people (Barrett & Keil 1996), like that he can only attend to one person or one event at the same time. External representations can anchor non-intuitive concepts on a more permanent basis. Seeing or reading them facilitates the recall of previously stored knowledge by individual mathematicians. Once mathematical concepts are nested outside of the brain, their evolution and cultural transmission is less vulnerable to corruption by individual mathematicians or to competition from ideas that are easier to learn, that speak more to the imagination, or that pose less computational demands. They gain a degree of autonomy that would be impossible to attain were they represented in the mind alone.

The consistent striving of European mathematicians to externalize operations – especially seemingly absurd ones – has given it a unique computational and representational power. This 'unreasonable' efficiency of mathematics as an epistemic tool is, I argue, an emergent property of its hybrid structure (Clark 2005). Since mathematicians can express operations that can never be performed in the mind alone (e.g. a negative number, which results from the subtraction of a larger number from a smaller one), mathematics can convey a range of ideas that are actually opaque. Let us examine this distinction between transparent and opaque more closely. A transparent concept is one to which we have semantic access; we intuitively grasp its meaning. In mathematics, Arabic numerals which denote positive integers {1, 2, 3…} are transparent concepts. Studies of brain activation in adults and even five-year-olds have shown that a number comparison task with Arabic numerals (say '5' and '3') activates the same brain areas as one that involves arrays of dots (say five dots and three dots) (Temple & Posner 1998); also the speed of computation is identical. Thus, the brain immediately translates a positive integer into a mental representation of its quantity. In contrast, when confronted with i or -3, no such translation takes place, because these operations cannot actually be performed in the mind (it is impossible to imagine a negative quantity). The only way to represent them is through an external set of symbols, the meaning of which remains semantically inaccessible to us. Through externalization, we expand our representational abilities by delegating operations that are impossible to perform in the brain to the world (by denoting them through a symbolic system). Mathematical concepts, in themselves a hybrid structure composed of transparent and non-transparent symbolic notations, can in its turn become an epistemic tool. We can denote processes in nature through mathematical formulations, because the latter have no clear semantic content. Therefore, identical functions and equations can be used in disparate contexts, including physics, biology, and anthropology.

6. Conclusion: implications of externalist and evolutionary approaches to mathematical cognition and the philosophy of mathematics

Any complex phenomenon can be studied successfully on different levels. Thus, we have sociological perspectives on mathematics, that look at the socio-cultural background in which mathematics emerges, and – more recently – approaches from cognitive science that look at the cognitive processes involved in mathematical thought. I am committed to psychoneural monism, the idea that human thought can be traced back to a materialist basis (Cartwright

2000: 4). Even if mathematics would have an existence outside of our brains and written sources (as is argued by mathematical realists), the only mathematics we could ever know is created by human beings, equipped with a human brain (Lakoff & Núñez 2000). And yet it is obvious that our evolved number sense, even when combined with other specialized neural systems including spatial and linguistic skills, cannot give rise to the complexities of mathematical thought. There are limitations to what the human mind is capable of imagining: negative quantities, non-Euclidean spaces and negative roots are unimaginable entities. Once humans became capable of performing cognitive actions in the world instead of in the mind alone, however, this limitation was lifted. It became possible to represent unimaginable operations. These flexible hybrid structures could in turn, because of their opaqueness and lack of semantic content, serve as epistemic tools in the sciences.

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Epistemic action, we suggest, demands spread of epistemic credit. If, as we confront some task, a part of the world functions as a process which, were it done in the head, we would have no hesitation in recognizing as part of the cognitive process, then that part of the world is (so we claim) part of the cognitive process.Â Language may be an example. Language appears to be a central means by which cognitive processes are extended into the world. Think of a group of people brainstorming around a table, or a philosopher who thinks best by writing, developing her ideas as she goes. It may be that language evolved, in part, to enable such extensions of our cognitive resources within actively coupled systems. A mathematical symbol is a figure or a combination of figures that is used to represent a mathematical object, an action on mathematical objects, a relation between mathematical objects, or for structuring the other symbols that occur in a formula. As formulas are entierely constitued with symbols of various types, many symbols are needed for expressing all mathematics.